Proceedings of
THE 10TH ASIA-PACIFIC ENGINEERING RESEARCH FORUM ON MICROWAVES AND ELECTROMAGNETIC THEORY

October 31-November 1, 2014
Fukuoka Institute of Technology
Fukuoka, Japan

Table of Contents

Determination of Generalized Scattering Matrix of Antenna from Characteristic Modes

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Abstract: A method for determining a generalized scattering matrix of an antenna using characteristic modes is presented here. Once characteristic currents at a feeding port, radiation patterns of characteristic currents and eigenvalues are known, the generalized scattering matrix can be calculated. It is shown that an antenna whose behavior is dominated by a single characteristic mode can be a canonical minimum scattering antenna. The formula is verified using a one and a half wavelength dipole antenna.

1. INTRODUCTION

The generalized scattering matrix of an antenna is widely used for an antenna analysis. For example, the generalized scattering matrix is useful for calculating a mutual coupling among antennas [1]–[3]. One method of determining the scattering matrix is to compute the amplitudes of scattered spherical waves independently for each incident spherical wave. Rubio et al. calculated the generalized scattering matrix from the generalized admittance matrix obtained using the segmentation-finite-element-Lanczos-Pade methodology [3].

The theory of characteristic modes was first developed by Garbacz [4] and was later refined by Harrington and Mautz [5]–[7]. The theory of characteristic modes is helpful in the analysis of electromagnetic scattering problems. In this paper, we derive a new formula that calculates a generalized scattering matrix of an antenna using the theory of characteristic modes.

The canonical minimum scattering (CMS) antenna is one which does not scatter electromagnetic fields when its feeding ports are open circuited [8]. Rogers demonstrates qualitatively that lossless antennas whose scattering behavior is dominated by a single characteristic mode can be CMS antennas [9]. In this paper, we find the condition where lossless or lossy antennas become CMS antennas using the derived formula.

Finally, we validate the formula by calculating the scattering matrix of a one and a half wavelength dipole antenna numerically.

2. Spherical Wave and Generalized Scattering Matrix

Electric fields and magnetic fields due to a current source $\mathbf{J}$ in free-space can be expressed as a superposition of spherical waves:

$$\mathbf{E} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} Q_{s,m,n}^{(c)} \mathbf{E}_{s,m,n}^{(c)} \tag{1a}$$

$$\mathbf{H} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} Q_{s,m,n}^{(c)} \mathbf{H}_{s,m,n}^{(c)} \tag{1b}$$

In this paper, the spherical wave function and ordering of spherical waves are the same as those used in the EM simulator FEKO [10]. The $e^{j\omega t}$ time convention is used throughout. $s = 1$ denotes TE mode and $s = 2$ denotes TM mode. $c = 1$ denotes standing waves, $c = 3$ denotes incoming waves, and $c = 4$ denotes outgoing waves. The relationship among standing wave, incoming wave, and outgoing wave is

$$\mathbf{E}_{s,m,n}^{(1)} = \frac{1}{2}\mathbf{E}_{s,m,n}^{(3)} + \frac{1}{2}\mathbf{E}_{s,m,n}^{(4)} \tag{2a}$$

$$\mathbf{H}_{s,m,n}^{(1)} = \frac{1}{2}\mathbf{H}_{s,m,n}^{(3)} + \frac{1}{2}\mathbf{H}_{s,m,n}^{(4)}. \tag{2b}$$

The complex amplitude of outgoing spherical wave, $Q_{s,m,n}^{(4)}$, is determined from current distribution using the following formula [2, p. 333]

$$Q_{s,m,n}^{(4)} = -(-1)^m \iiint \mathbf{E}_{s,-m,n}^{(1)} \cdot \mathbf{J} dv \tag{3}$$

where $\mathbf{J}$ is current density.

Amplitudes of incident and reflected waves on the feed waveguide and amplitudes of incoming and outgoing spherical waves are related by the following generalized scattering matrix equation [2]:

$$\begin{bmatrix} w \\ \hline \mathbf{b} \end{bmatrix} = \left[\begin{array}{c|c} \Gamma & \mathbf{R} \\ \hline \mathbf{T} & \mathbf{S} \end{array}\right] \begin{bmatrix} v \\ \hline \mathbf{a} \end{bmatrix} \tag{4}$$

where v is the amplitude of the incident wave at an antenna's feeding port and w is the amplitude of the reflected wave at the feeding port. $\mathbf{a}$ is an infinite dimensional column matrix containing amplitudes of incoming spherical waves and $\mathbf{b}$ is an infinite dimensional column matrix containing amplitudes of outgoing spherical waves. Γ is a reflection coefficient and $\mathbf{T}$, $\mathbf{R}$, and $\mathbf{S}$ describe, respectively, the transmitting, receiving, and scattering properties of antennas. The wave functions are normalized such that one half of the square of the absolute value of the amplitude is the power carried by the wave.

The receiving pattern of a reciprocal antenna can be found from the transmitting pattern. **R** is determined using the following equation [2, p. 36]:

$$R_{s,m,n} = (-1)^m T_{s,-m,n} \quad (5)$$

where $T_{s,m,n}$ and $R_{s,m,n}$ are the element of **T** and **R**, respectively.

3. Theory of Characteristic Modes

Although the theory of characteristic modes is extensively described in [5]–[7], we will summarize the theory of characteristic modes briefly. The eigenvalue equation defining the characteristic modes is

$$X(\mathbf{J}_n) = \lambda_n R(\mathbf{J}_n) \quad (6)$$

There are two types of eigenvalue equation when bodies are lossy [7]. In this paper, we choose the eigenvalue equation so that eigenvalues and eigenvectors are real. R and X in (6) are, respectively, the real and imaginary part of the impedance operator in the method of moments. The eigenvector $\mathbf{J}_n$ is called the characteristic current. The characteristic currents are normalized such that

$$\iiint \mathbf{J}_n \cdot R(\mathbf{J}_n) dv = 1 \quad (7)$$

The characteristic modes are ordered according to $|\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \cdots$. Total current on an antenna can be represented by the linear combination of all characteristic currents:

$$\mathbf{J} = \sum_{n=1}^{\infty} \frac{V_n^i \mathbf{J}_n}{1 + j\lambda_n} \quad (8)$$

where V_n^i is the modal excitation coefficient, and determined by

$$V_n^i = \iiint \mathbf{J}_n \cdot \mathbf{E}^i dv \quad (9)$$

where $\mathbf{E}^i$ is the incident electric field.

4. IV. FORMULA FOR GENERALIZED SCATTERING MATRIX OF ANTENNAS

Fig. 1 represents the antenna that is addressed in this paper. It is assumed that the material body of the antenna is composed of conductors or dielectrics, and is linear and reciprocal. The material body can be lossy. It is also assumed that the number of feeding ports is one and the antenna is excited at an infinitesimal gap on the conducting wire.

We first consider the scattering matrix of bodies without a feeding port, where the infinitesimal gap is filled with a conductor. In this case, w, v, $\mathbf{\Gamma}$, **T**, and **R** in (4) are eliminated and **a**, **b**, and **S** remain. Since

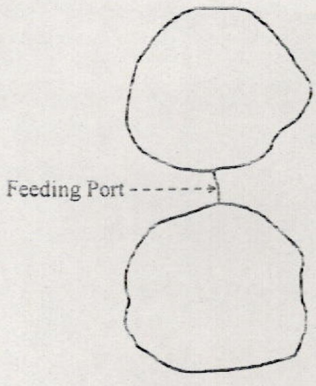

Fig. 1. Material body and feeding port of antennas

the incident wave is finite at the coordinate origin, the incident wave is expanded in terms of the standing spherical waves. Let the standing spherical wave of index (s, m, n) be incident on the body and its amplitude be $2a_{s,m,n}$. In other words, $2a_{s,m,n}\mathbf{E}^{(1)}_{s,m,n} \left(= a_{s,m,n}\mathbf{E}^{(3)}_{s,m,n} + a_{s,m,n}\mathbf{E}^{(4)}_{s,m,n}\right)$, and $2a_{s,m,n}\mathbf{H}^{(1)}_{s,m,n} \left(= a_{s,m,n}\mathbf{H}^{(3)}_{s,m,n} + a_{s,m,n}\mathbf{H}^{(4)}_{s,m,n}\right)$ are incident on the antenna. Let the amplitude of the outgoing spherical wave of index (s, m, n) generated by pth characteristic current be, $T^p_{s,m,n}$ i.e.

$$T^p_{s,m,n} = -(-1)^m \iiint \mathbf{J}_p \cdot \mathbf{E}^{(1)}_{s,-m,n} dv \quad (10)$$

Using (9), the modal excitation coefficient of pth characteristic current is given by

$$V_p^i = \iiint \mathbf{J}_p \cdot 2a_{s,m,n}\mathbf{E}^{(1)}_{s,m,n} dv \quad (11)$$

Comparing (11) with (10), we notice that (11) is same as $-2(-1)^m T^p_{s,-m,n} a_{s,m,n}$. When multiple spherical waves are incident, the amplitudes of the incoming and outgoing spherical waves are to be represented by a column matrix **a**. Then, the modal excitation coefficient of pth characteristic current is

$$V_p^i = -2\mathbf{R}^p \mathbf{a} \quad (12)$$

where $\mathbf{R}^p$ is a row matrix and the element of $\mathbf{R}^p$ is defined by

$$R^p_{s,m,n} = (-1)^m T^p_{s,-m,n} \quad (13)$$

When incoming and outgoing waves with amplitude **a** are incident, the outgoing spherical wave amplitudes generated by pth characteristic current with the weighting coefficient is

$$\mathbf{b}^p_{scattered} = \frac{-2}{1 + j\lambda_p} \mathbf{T}^p \mathbf{R}^p \mathbf{a} \quad (14)$$

where the elements of $\mathbf{T}^p$ is given by $T^p_{s,m,n}$. Therefore, the amplitudes for the outgoing spherical waves of total field (scattered field + incident field) when **a** is incident is

$$\mathbf{b} = \mathbf{a} - 2\sum_{p=1}^{\infty} \frac{1}{1+j\lambda_p} \mathbf{T}^p \mathbf{R}^p \mathbf{a} \quad . \quad (15)$$

Next, consider the scattering matrix of antennas with a feeding port. A short circuited antenna can be considered as a body without a port. If a port is short circuited,

$$w = -v \quad . \quad (16)$$

Substituting (16) into (4) and manipulating, we have

$$\mathbf{b} = \left(-\frac{1}{1+\Gamma} \mathbf{TR} + \mathbf{S} \right) \mathbf{a} \quad . \quad (17)$$

Comparing (17) with (15), we have

$$\mathbf{S} = \mathbf{I} + \frac{1}{1+\Gamma} \mathbf{TR} - 2\sum_{p=1}^{\infty} \frac{1}{1+j\lambda_p} \mathbf{T}^p \mathbf{R}^p \quad (18)$$

where **I** is a unit matrix.

If we excite an infinitesimal gap of an antenna with V volts, the modal excitation coefficient for pth characteristic current is, by (9),

$$V_p^i = VI_p \quad (19)$$

where I_p is a current at the feeding port for pth characteristic current. Therefore, from (8) the current density on the antenna when V volts is applied at the port is

$$\mathbf{J} = \sum_{p=1}^{\infty} \frac{VI_p}{1+j\lambda_p} \mathbf{J}_p \quad . \quad (20)$$

In this case, the outgoing spherical wave amplitudes that the antenna transmit is

$$\mathbf{b}_{transmitted} = \sum_{p=1}^{\infty} \frac{VI_p}{1+j\lambda_p} \mathbf{T}^p \quad . \quad (21)$$

If a wave with an amplitude of 1 is incident at a feeding port, the voltage at the gap is $V = \sqrt{Z_0}(1+\Gamma)$ where Z_0 is the characteristic impedance of the waveguide of an antenna. Here, the reference plane is at the feeding gap and only one mode exists in the waveguide. Therefore, the modal transmitting pattern of the antenna is

$$\mathbf{T} = \sum_{p=1}^{\infty} \frac{\sqrt{Z_0}(1+\Gamma)I_p}{1+j\lambda_p} \mathbf{T}^p \quad . \quad (22)$$

The reflection coefficient is also can be determined from the characteristic currents. From (20), the current at the feeding port is

$$I = \sum_{p=1}^{\infty} \frac{V(I_p)^2}{1+j\lambda_p} \quad . \quad (23)$$

Therefore, the input admittance of the antenna is

$$Y_{in} = \frac{1}{Z_0} \frac{1-\Gamma}{1+\Gamma} = \sum_{p=1}^{\infty} \frac{(I_p)^2}{1+j\lambda_p} \quad (24)$$

and from this equation we obtain the reflection coefficient.

The generalized scattering matrix can be obtained from the characteristic currents at a feeding port, transmitting patterns of characteristic currents in terms of spherical waves, and eigenvalues.

5. Canonical Minimum Scattering Antenna

In this section, we will find the condition where an antenna becomes a CMS antenna. We assume that the eigenvalues of higher order modes are very large compared with the eigenvalue of the lowest order mode so that the currents on the antenna are dominated by a single characteristic current. In this case, **T** becomes

$$\mathbf{T} \approx \frac{\sqrt{Z_0}(1+\Gamma)I_1}{1+j\lambda_1} \mathbf{T}^1 \quad , \quad (25)$$

R becomes, from (5),

$$\mathbf{R} \approx \frac{\sqrt{Z_0}(1+\Gamma)I_1}{1+j\lambda_1} \mathbf{R}^1 \quad , \quad (26)$$

and **S** becomes

$$\mathbf{S} \approx \mathbf{I} + \frac{1}{1+\Gamma} \mathbf{TR} - \frac{2}{1+j\lambda_1} \mathbf{T}^1 \mathbf{R}^1 \quad . \quad (27)$$

Substituting (25) and (26) into (27) and rearranging,

$$\mathbf{S} \approx \mathbf{I} + \frac{1}{1+\Gamma} \mathbf{TR} - \frac{2(1+j\lambda_1)}{Z_0(1+\Gamma)^2 (I_1)^2} \mathbf{TR} \quad . \quad (28)$$

From (24), the input impedance of the antenna is

$$Z_{in} = Z_0 \frac{1+\Gamma}{1-\Gamma} \approx \frac{1+j\lambda_1}{(I_1)^2} \quad . \quad (29)$$

Substituting (29) into (28) and rearranging, we have

$$\mathbf{S} = \mathbf{I} - \frac{1}{1-\Gamma} \mathbf{TR} \quad . \quad (30)$$

According to Rubio and Izquierdo [11, eq. (16)], this is the same as the **S** of CMS antennas. Therefore, an antenna whose behavior is dominated by a single characteristic mode can be a CMS antenna.

6. Validation

To verify the theory, we calculated the **S** matrix in (4) by two different methods and compared the two results. In one method, we did simulations where each spherical wave was incident on the antenna with a load and found the amplitudes of scattered spherical waves generated by the induced current on the antenna. From this we determined the scattering matrix. In another method, we calculated a scattering matrix from characteristic modes using the formula presented in this paper.

The antenna used in the simulation is a dipole antenna. The length of the dipole antenna was 300 mm and diameter of wire 1 mm. The conductivity of the wire was 10^5 S/m. The antenna was fed at its center and the characteristic impedance of the waveguide was 150 Ω. The simulated frequency was 1.5 GHz, giving the dipole length of 1.5 wavelengths. The wire is on the z axis and the center of the dipole was located on the origin of the coordinate system.

We calculated the impedance matrix by using the pulse function for the expansion and point matching for testing [12]. When we calculated the S matrix by the first method, we terminated 150 Ω at the feeding port. In the method presented, we calculated the S matrix using (18) and (22) with five dominant characteristic modes. When we calculated the reflection coefficient, we used many characteristic modes because if we used only a few characteristic modes we would not be able to compute the input admittance exactly.

Table I shows several characteristic modes for which the eigenvalues are small and the dominant spherical waves that each characteristic mode generates. Fig. 2 (a) shows the S matrix calculated by the first method and Fig. 2 (b) shows the **S** matrix calculated by the method presented. We show the dominant spherical waves in Fig. 2. Notice they are almost exactly same and it shows the validity of the proposed method to obtain the antenna scattering matrix. If we use more characteristic modes, the error between the two matrices becomes smaller.

TABLE I
Eigenvalue of Characteristic Mode and Dominant Spherical Wave

Mode Number	Eigenvalue	Dominant Spherical Wave	Amplitude
1	0.447	TM_{01}	-0.243
		TM_{03}	0.878
		TM_{05}	0.265
2	2.532	TM_{02}	0.926
		TM_{04}	0.306
3	2.612	TM_{01}	-0.950
		TM_{03}	-0.261
4	-23.638	TM_{02}	0.215
		TM_{04}	-0.755
		TM_{06}	-0.190
5	-128.395	TM_{05}	-0.391

$$\begin{matrix} TM_{01} \\ TM_{02} \\ TM_{03} \\ TM_{04} \\ TM_{05} \end{matrix} \begin{bmatrix} 0.6886+j0.4520 & 0 & 0.1061+j0.3096 & 0 & 0.0463+j0.0756 \\ 0 & 0.7682+j0.5824 & 0 & -0.0759+j0.2073 & 0 \\ 0.1061+j0.3096 & 0 & 0.2444+j0.2764 & 0 & -0.2259+j0.0599 \\ 0 & -0.0759+j0.2073 & 0 & 0.9727+j0.0155 & 0 \\ 0.0463+j0.0756 & 0 & -0.2259+j0.0599 & 0 & 0.9317+j0.0091 \end{bmatrix}$$

(a)

$$\begin{matrix} TM_{01} \\ TM_{02} \\ TM_{03} \\ TM_{04} \\ TM_{05} \end{matrix} \begin{bmatrix} 0.6881+j0.4514 & 0 & 0.1065+j0.3105 & 0 & 0.0464+j0.0760 \\ 0 & 0.7682+j0.5824 & 0 & -0.0759+j0.2072 & 0 \\ 0.1065+j0.3105 & 0 & 0.2442+j0.2754 & 0 & -0.2260+j0.0594 \\ 0 & -0.0759+j0.2072 & 0 & 0.9727+j0.0158 & 0 \\ 0.0464+j0.0760 & 0 & -0.2260+j0.0594 & 0 & 0.9316+j0.0092 \end{bmatrix}$$

(b)

Fig. 2. **S** of the dipole antenna obtained by two methods (a) first method (b) presented method

7. Conclusion

We showed the full generalized scattering matrix can be calculated once characteristic currents at a feeding port, transmitting patterns of characteristic currents in terms of spherical waves, and eigenvalues are known. From the formula presented in this paper, we demonstrated that an antenna whose behavior is dominated by a single characteristic mode can be a CMS antenna.

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