# An iterative FEM with fast multipole updates for scattering from an electically large object

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## 1. Introduction

As known widely, the finite element method(FEM) has been widely used as a powerful tool for solving numerous electromagnetic problems since it is able to deal with arbitrary geometries and inhomogeneous and complex materials. Besides, the FEM gets used for solving scattering problems as well as typical bounded problems through hybridization with other methods such as a boundary integral method(BIM) or incorporation with an appropriate absorbing boundary condition. Among these methods for applying the FEM to scattering problems, the iterative FEM has been proposed[1-2] and complemented[3] by eliminating the internal resonance[4] in the recent years. According to this method, the FEM gives an efficient and accurate result with only a small number of meshes near around a scatterer through several iterative updates of the boundary conditions. The generated system matrix does not only preserve sparsity, which is a very good property of the typical FEM, but is also invariant during the iterations. Besides, the matrix for updating the boundary conditions is invariant as well. Thus, the numerical calculation for these two matrices is performed only once at the first iteration. These properties make this method efficient and competitive with other methods such as a finite element – boundary integral method(FEBIM).

However, this method still has a bottleneck that the updating matrix is a full matrix. Just as the method of moments, this property makes it difficult to apply the iterative FEM to scattering by an electrically large object. Assuming that the numbers of unknowns on the fictitious boundary and on the boundary where the equivalent current is calculated are $N_F$ and $N_M$, respectively, both the operation counts for matrix-vector multiplication per iteration and the memory requirements for storage of the updating matrix will be proportional to $O(N_F N_M)$. If $N_F$ is almost equal to $N_M$, they will be $O(N_F^2)$.

Therefore, in this paper, in order to apply the iterative FEM to a large-body problem, we develop a scheme to reduce both the operation counts and the memory requirements to a lower order by applying the well known fast multipole method(FMM) to the boundary updating procedure.

**2. Thoery**

As a very simple example of this method, we analyzed scattering characteristics by a 2D large conducting cylinder and a 3D long crack recessed in the ground plane. In 2D case, the system is assumed to be excited by a TM polarized wave for brevity. As 2D and 3D basis functions, triangular and vector prism elements are used respectively. Fig.1 represents the geometries to be analyzed and the fictitious surfaces on which the boundary fields will be updated. Note that those fictitious surfaces $\Gamma_F$ and $A_2$ can be placed close considerably to the boundary surfaces $\Gamma_M$(or $\Gamma_C$) and $A_1$ where the equivalent currents are calculated. As mentioned above, in order to avoid the interior resonance-like phenomena, the radiation type boundary condition on the fictitious surfaces should be applied as follows.

$$\frac{\partial \phi}{\partial n} + jk_0\phi = \psi \quad \text{on } \Gamma_F \text{ (in 2D TM case)}$$

$$\hat{n}\times\nabla\times\mathbf{E} + jk_0\hat{n}\times\hat{n}\times\mathbf{E} = \mathbf{U} \text{ on } A_2 \text{ (in 3D case)} \tag{1}$$

This mixed boundary condition means that the left side of the Sommerfeld radiation condition does not vanish in the near field region. The key idea of an iterative FEM is based on the update of this residual term. With these mixed boundary conditions, the functionals of 2D and 3D problems are given by

$$F(\phi) = \frac{1}{2}\int_R \frac{1}{\mu_r}\nabla\phi\cdot\nabla\phi - k_0^2\varepsilon_r\phi^2\, ds + \frac{jk_0}{2}\int_{\Gamma_F}\phi^2\, ds - \int_{\Gamma_F}\phi\psi\, ds \tag{2}$$

$$F(\mathbf{E}) = \frac{1}{2}\int_V \frac{1}{\mu_r}(\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{E}) - k_0^2\varepsilon_r\mathbf{E}\cdot\mathbf{E}\, dv + \frac{jk_0}{2}\int_{A_2}(\hat{n}\times\mathbf{E})\cdot(\hat{n}\times\mathbf{E})\, ds + \int_{A_2}\mathbf{E}\cdot\mathbf{U}\, ds \quad .$$

Initially, $\psi$ and **U** are calculated from the equation (1) on the assumption that $\phi$ and **E** are the same as the incident electric field $\phi_{inc}$ and $\mathbf{E}_{inc}$, respectively. Using these initial values with the functionals in the equation (2) minimized by the typical FEM procedure, the fields can be determined everywhere. From these calculated fields, the equivalent currents on the boundary $\Gamma_m$(or $\Gamma_C$) and $A_1$ are introduced using the equivalence theorem, and these equivalent sources generate the new scattered fields on the radiation boundary surfaces using the free space Green's function.

In this way, the fields on the radiation boundary are updated again, and this means the iterative update of the residual term $\psi$ and **U** in the equation (1). Through a few iterations of this procedure, the fields converge to the exact values.

However, there are two time-consuming factors to overcome in this method. First, since standard FEM procedures are repeated several times, each iteration step requires inversion of a system matrix. But if the point that the sparse system matrix does not change during iterations is fully exploited in the implementation, computational loads can be greatly reduced. Second, numerical integration of the Green's function is required whenever the fields on the radiation boundary surfaces are updated. Usually, this step is regarded as a time-consuming procedure. However, once numerical integration of the Green’s function is performed at the first iteration, the result can be used in the subsequent iterations. Therefore, the subsequent iterations do not impose additional computational loads. These two properties can make this method efficient.

However, it is clear that the efficiency of this method is limited to the analysis of an electrically small scatterer. The system matrix is highly sparse and its bandwidth can be reduced greatly by using several bandwidth reduction schemes. CPU time and memory requirements for solving this sparse system are able to be made $O(N)$ by fixing the matrix bandwidth constant irrespective of increase of the number of unknowns. This can be achieved since it is found that the distance between the radiation boundary surface and the equivalent source surface can be preserved nearly constant regardless of the electrical size of a scatterer. Therefore, the procedure to solve the sparse system is not a problem even though the number of unknowns increases. On the contrary, the procedure for improving the fields on the radiation boundary generates a full matrix based on the integral equations. Thus, both memory and CPU time requirements for matrix-vector multiplication will be proportional to an order of the product of the numbers of unknowns on two boundaries. This property makes it difficult to apply the iterative FEM to a large-body problem.

In this paper, we show that this updating procedure for a large-body problem can be greatly improved by using the fast multipole method(FMM) to reduce operation counts and memory requirements to a lower order. By a typical fast multipole procedure, the boundary elements on two boundaries are subdivided into several groups, respectively. Since the key technique in the FMM is based on decomposing the interaction between two far groups into three parts, that is, aggregation, translation, and disaggregation by using the addition theorem[5-6], applying it to the iterative FEM, the updated fields on the fictitious boundary between far groups can be calculated efficiently by separating the group-to-group interactions from the inter-group interactions in the same manner. Therefore, the total CPU time and memory requirements for updating the boundary fields are reduced to about $O(N_F^{1.5})$ with the one-level fast multipole scheme. In addition, further improvements will be achieved applying the multi-level scheme to this updating procedure.

### 3. Numerical Results

As mentioned above, in order to examine the validity and capability of the proposed method, we analyzed the scattering by a circular cylinder and a cavity-backed aperture as the 2D and 3D examples, respectively. In case of a circular cylinder, since an analytic series solution exists, the result obtained from this extended iterative FEM is compared with the series solution. For a circular cylinder of 3λ radius with a TM plane-wave excitation, the magnitude and phase of the electric current on the conducting surface and bistatic radar cross section(RCS) patterns are illustrated in Fig.2. Also, Fig. 3 shows the monostatic RCS patterns of a 3D long crack recessed in the ground plane, and they are compared with the conventional FEBIM. As shown in two figures, very good agreements are made between the results. Fig. 4 represents the total computation time for analyzing these structures in order to verify the efficiency of the proposed method. From these results, this method is shown to be more efficient than other conventional ones

**4. Conclusions**

An efficient iterative FEM is extended and applied to the scattering by an electrically large object. With the hybridization of the well-known fast multipole scheme, we show that the proposed method is accurate and can be more efficient than the conventional fast methods. As well as a conducting cylinder and a cavity-backed aperture, this extended iterative FEM is expected to be able to deal with many scattering problems of a wide class of electrically large objects.

**[Reference]**

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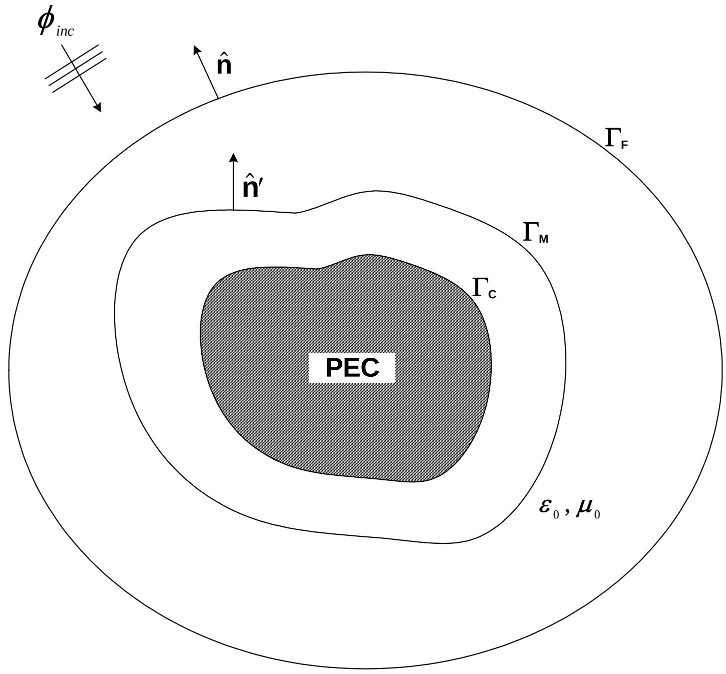
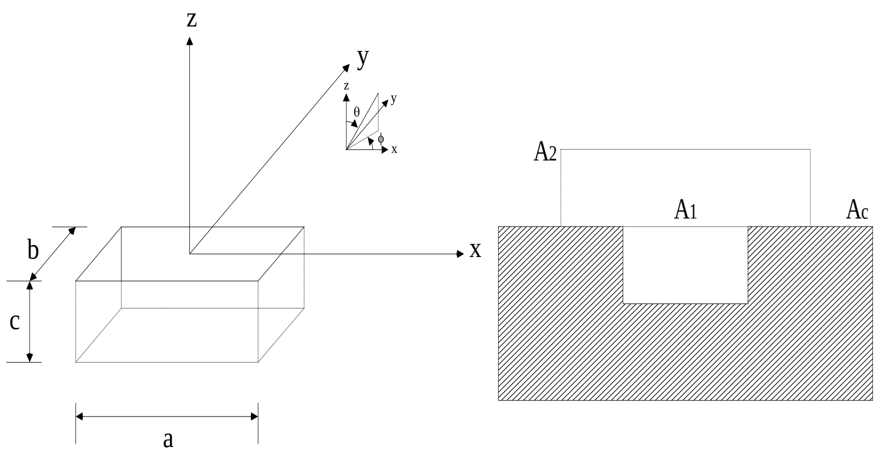


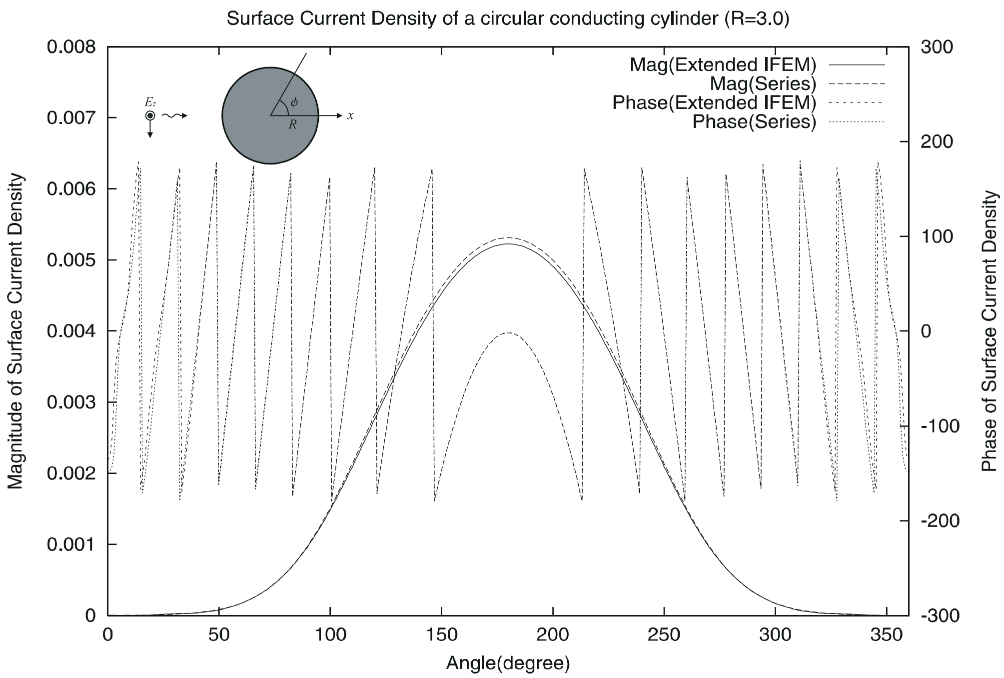

(a)




(b)

Fig. 1. Geometry and fictitious boundary surfaces

(a) 2D conducting cylinder (b) 3D cavity-backed aperture

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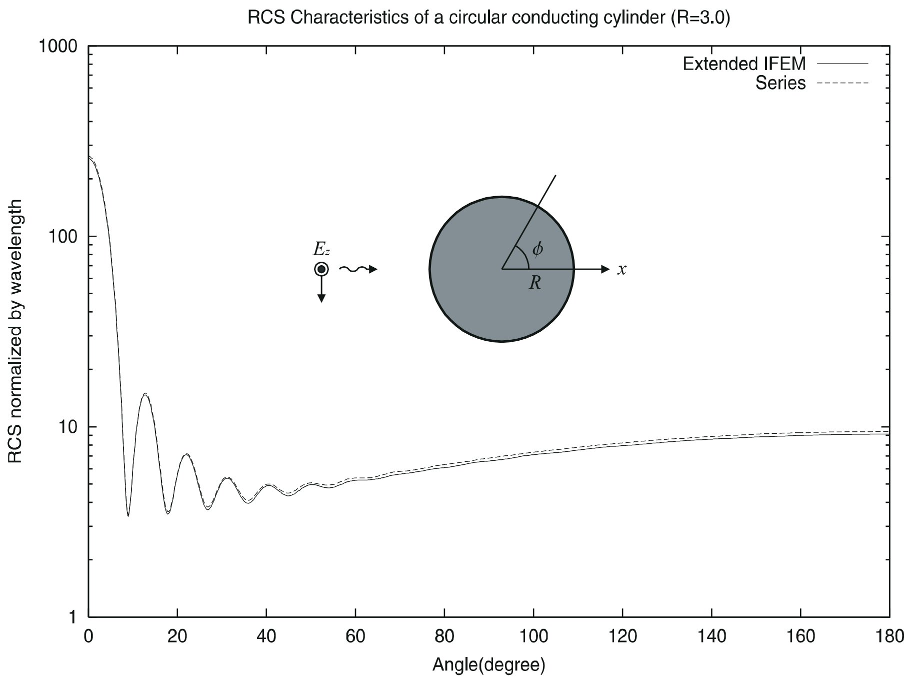
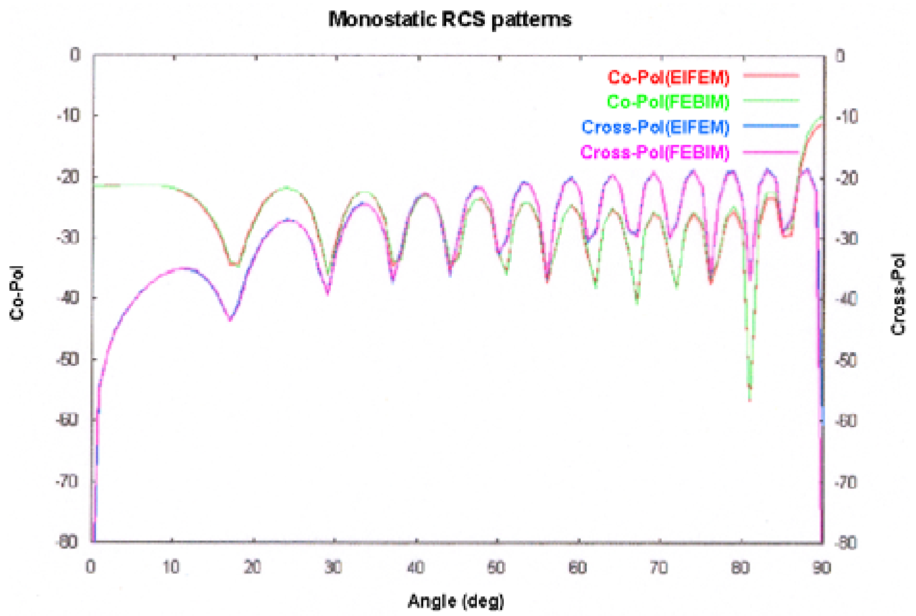



(b)

Fig. 2 (a) Surface current density and (b) bistatic RCS patterns when R = 3.0λ

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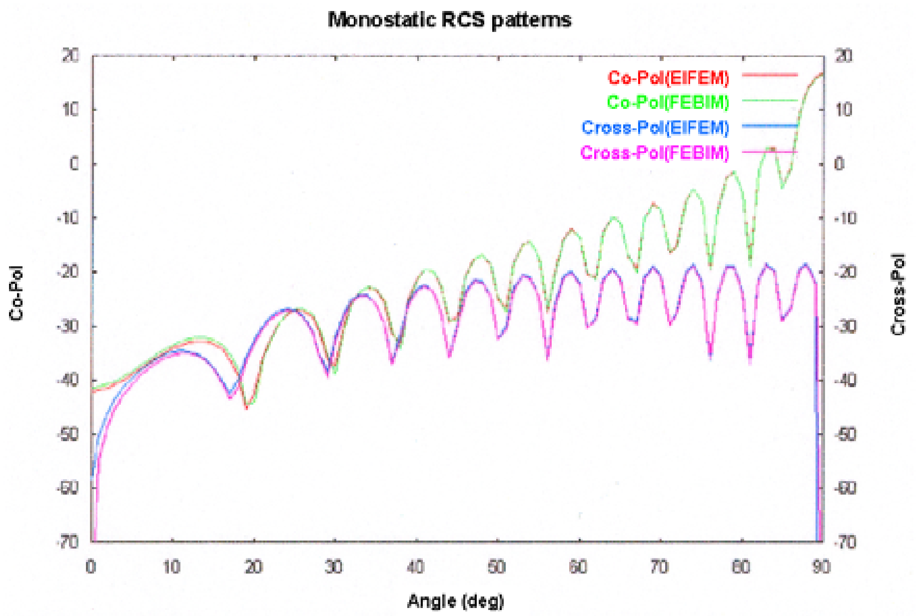
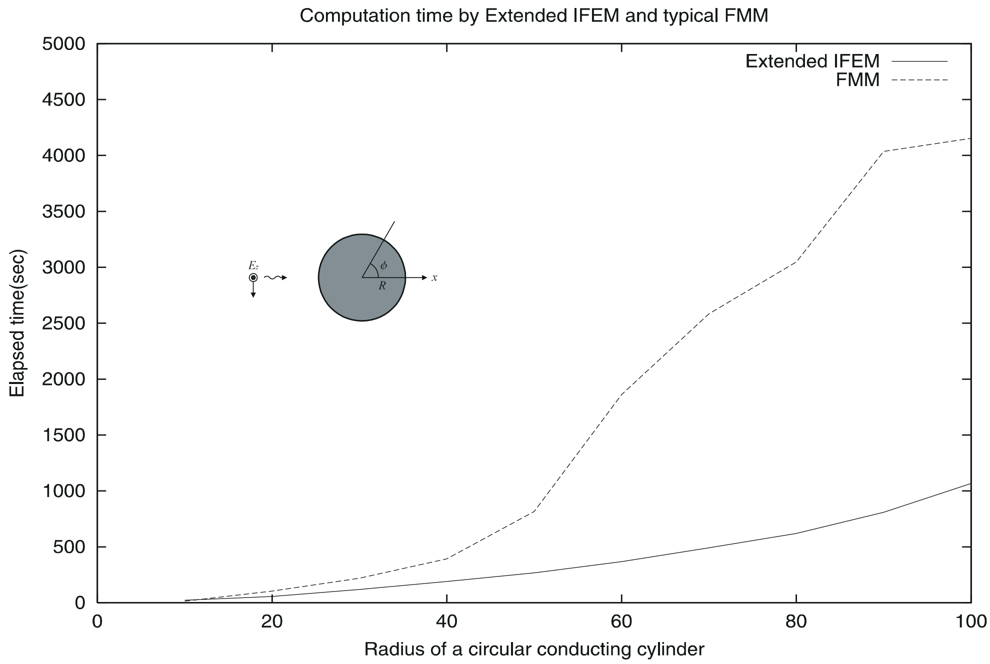



(b)

Fig. 3 Monostatic RCS patterns of a 3D long crack recessed in the ground plane ( a=10λ, b=0.1λ, c=0.1λ, θ=40 ˚ )

(a) polarization angle α : 90 ˚ (b) polarization angle α : 0 ˚

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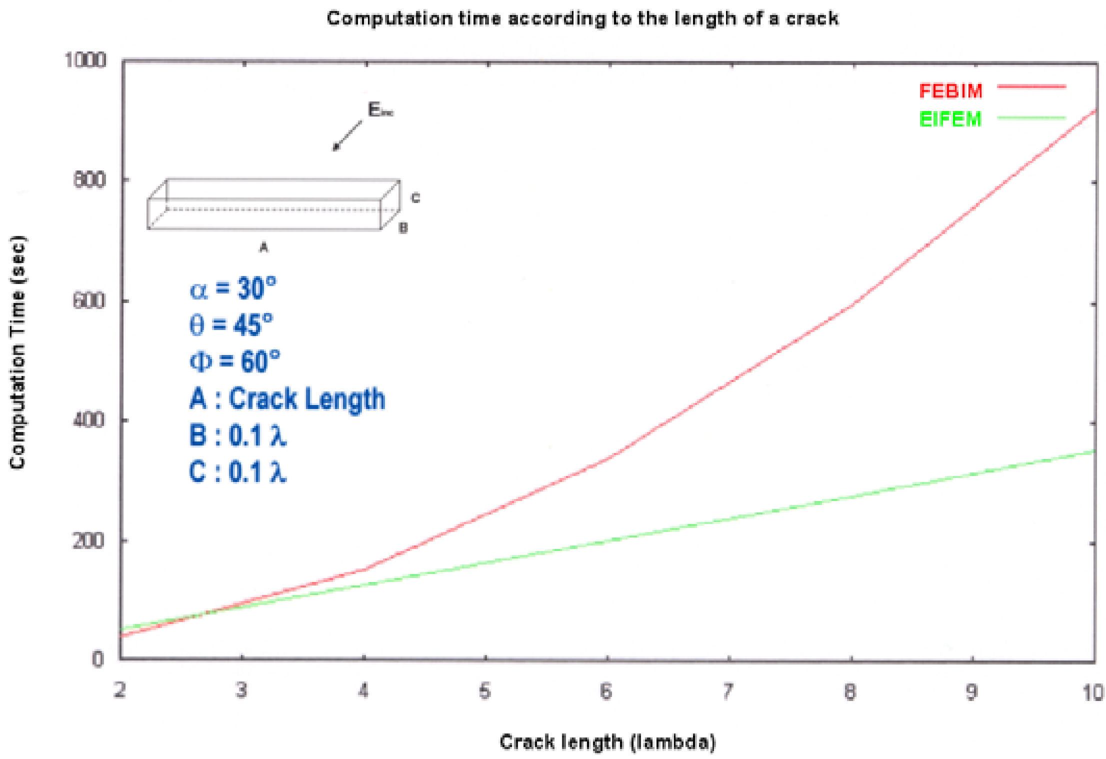


(b)

Fig. 4 Comparison of total computation time between the extended iterative FEM and other methods

(a) 2D case (comparison with conventional FMM)

(b) 3D case (comparison with conventional FEBIM)